On time evolutions associated with the nonstationary Schrödinger equation

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Abstract

The set of integrable symmetries of the nonstationary Schrödinger equation is shown to admit a natural decomposition into subsets of mutually commuting symmetries. Hierarchies of time evolutions associated with each of these subsets ultimately lead to nonlinear (possibly, operator) equations of the Kadomtsev–Petviashvili I type or its higher analogues, thus demonstrating that the linear problem itself constructively determines the associated nonlinear integrable evolution equations and their hierarchies.

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1 Introduction

In this article, we use the nonstationary Schrödinger (NS) equation as an example to demonstrate that a given linear problem determines the nonlinear evolutions associated with it. Our approach is based on the study of the set of symmetries for the NS equation using the inverse scattering transform (IST). We here consider those symmetries that can be called integrable, which means that the associated flows can be expressed in terms of the potential of the linear problem or in terms of some special values of the Jost solutions. The set of all integrable symmetries is decomposed into subsets of mutually commuting ones; with each of these subsets, we associate a hierarchy of evolution equations. We show that all these subsets of commuting symmetries have a natural hierarchy structure, and we investigate the corresponding evolutions. We find that the Jost solution of the original NS equation with respect to the two lowest times of any such hierarchy also obeys an NS equation with an operator-valued potential. Moreover, the third (or any higher) time of the corresponding hierarchy provides an operator differential equation whose linear part coincides with the linear part of the Kadomtsev–Petviashvili I (KPI) equation [1] (or of its higher analogue respectively). This linearized KPI equation naturally appears as the equation for the spectral data here; it follows from a simple commutator identity that the spectral data satisfy this equation (or its higher analogues) regardless of the choice of the subclass of commuting symmetries. We also show that there are two distinguished subclasses of commuting flows characterized by the property that the above-mentioned operator potential is just a multiplication operator. These subsets of symmetries correspond to the standard evolutions and the ones that can be considered dual to them. We consider these subsets in detail and show that the second one leads to a transformation of both the dependent and independent variables, which essentially extends the range of applicability of the IST.

The study of different symmetries associated with the NS equation (or other linear problems) goes back to the dressing procedure [2, 3] and was investigated either by these methods and their extensions or in formal algebraic terms (see [4]–[9]). The approach adopted below is peculiar: we work strictly in the IST framework [10]–[13], starting with the NS equation with respect to (x_1, x_2) variables with a real potential $u(x_1, x_2)$ assumed to be a smooth function that rapidly decays at infinity and satisfies some weak conditions on its norm [14] to guaranty the solvability of the direct and inverse problems.

Our approach is based on a special version of the IST, the so-called resolvent approach (see [15]–[19]). In the next section, we briefly describe some constructions that enable us to combine the methods of the IST with the algebraic schemes of the type in [5]–[7]. To control the transformations of all the involved objects—spectral data, Jost solutions, and the potential—we use the so-called scattering theory on a nontrivial background [20]. In Sec. 3, we consider integrable symmetries in this framework and derive general results related to their commuting subsets (evolutions). The simplest examples of such evolutions are given in Sec. 4.

2 Extended operators and resolvent approach

The first step in the resolvent approach consists in a special extension of the differential operators. Precisely, any given differential operator A with the kernel

$$A(x, x') = A(x_1, x_2, \partial_{x_1}, \partial_{x_2})\delta(x_1 - x'_1)\delta(x_2 - x'_2) \tag{2.1}$$

is replaced with the differential operator $A(q)$ with the kernel

$$A(x,x';q)=e^{-q(x-x')}A(x,x')\equiv A(x_1,x_2,\partial_{x_1}+q_1,\partial_{x_2}+q_2)\delta(x-x'), \tag{2.2}$$

where $qx=q_1x_1+q_2x_2$ (we consider only the two-dimensional case here). In (2.2), q is a real (two-component) vector; below, we often omit explicit indication of the operator dependence on the variable q. The standard definitions of the dual operator, the complex conjugate operator, and the Hermitian conjugate operator are modified as follows:

$$\begin{aligned}&A^{\mathrm{dual}}(x,x';q)=A^{\mathrm{t}}(x',x;-q), \qquad A^{*}(x,x';q)=\overline{A(x,x';q)},\\ &A^{\dagger}(x,x';q)=\overline{A^{\mathrm{t}}(x',x;-q)},\end{aligned} \tag{2.3}$$

where the superscript t denotes transposition in the matrix case. In what follows, we refer to the kernel $A(x,x';q)$ of the operator A as the kernel in the x representation to distinguish it from the kernel of the same operator in the $(p,\mathbf{q})$ representation defined via the "shifted" Fourier transform,

$$A(p;\mathbf{q})=\frac{1}{(2\pi)^2}\int dx\int dx' e^{i(p+\mathbf{q}_{\Re})x-i\mathbf{q}_{\Re}x'}A(x,x';\mathbf{q}_{\Im}), \tag{2.4}$$

where p and $\mathbf{q}$ are respectively real and complex two-dimensional vectors. In these terms, the meaning of this extension of the kernels of differential operators can be clarified by observing that the kernels $A(p;\mathbf{q})$ of differential operators are polynomial functions of the variable $\mathbf{q}$. As shown in [15], this analyticity property of the extended differential operators considerably simplifies the study of their spectral transform.

Below, we consider integral operators $A(q)$ (A, for short) with kernels $A(x,x';q)$ belonging to the space $\mathcal{S}'$ of the Schwartz distributions with respect to all six real variables, x_1, x_2, x_1', x_2', q_1, and q_2 (or in the space of the $(p,\mathbf{q})$ variables with respect to (2.4)). We also consider a special subclass $\mathcal{M}\subset\mathcal{S}'$ of such operators, whose kernels are Schwartz distributions with respect to the variables x and x' and are piecewise continuous functions with respect to the variable q.

The relevance of this subclass is demonstrated by the following simple example. Let D_j denote the extension of the differential operator ∂_{x_j}, $j=1,2$, by (2.2), i.e.,

$$D_j(x,x';q)=(\partial_{x_j}+q_j)\delta(x-x'), \quad j=1,2. \tag{2.5}$$

This operator is uniquely invertible on the subclass $\mathcal{M}$. Its inverse is given by the kernel

$$D_j^{-1}(x,x';q)=\operatorname{sgn}q_j e^{-q_j(x_j-x_j')}\theta(q_j(x_j-x_j'))\delta(x_{j+1}-x_{j+1}'), \quad j=1,2, \tag{2.6}$$

($j+1$ is understood mod 2) and is exactly the standard resolvent of the operator ∂_{x_j}. This observation gave the name *resolvent approach* for the method based on the above extension of the original differential operators. The uniqueness of inverse (2.6) follows because the only (up to a factor) annulator of D_j belonging to $\mathcal{S}'$ has the kernel $\delta(q_j)\delta(x_{j+1}-x_{j+1}')$, which does not belong to $\mathcal{M}$. We also mention that if the q dependence of the extended differential operator can be trivially removed by multiplying the kernel by $\exp q(x-x')$ (cf. (2.2)), the q dependence of the inverse operator cannot be removed in this way, as is shown by the example in (2.6).

We turn to the NS operator

$$\mathcal{L}_x=i\partial_{x_2}+\partial_{x_1}^2-u(x_1,x_2). \tag{2.7}$$

Using (2.2), we can write its extension as

$$L = L_0 - U, \tag{2.8}$$

where L_0 is the extension corresponding to the zero-potential case and is equal to

$$L_0 = iD_2 + D_1^2 \tag{2.9}$$

by (2.5). In (2.8), we introduce the multiplication operator U with the kernel

$$U(x, x'; q) = u(x)\delta(x - x'). \tag{2.10}$$

The main object in our approach is the (extended) resolvent of operator (2.8) defined as the inverse of the operator L on the subclass $\mathcal{M}$, i.e.,

$$LM = ML = I, \tag{2.11}$$

where I is the unity operator,

$$I(x, x'; q) = \delta(x - x'). \tag{2.12}$$

We mention that the kernel of the resolvent M_0 of the zero-potential operator L_0 (2.9) is equal to

$$M_0(x, x'; q) = \frac{\operatorname{sgn}(x_2 - x_2')}{2\pi i} \int d\alpha\theta\Big((q_2 - 2\alpha q_1)(x_2 - x_2')\Big)e^{-[q+i\ell(\alpha+iq_1)](x-x')}, \tag{2.13}$$

where we introduce the special two-vector

$$\ell(\alpha) = (\alpha, \alpha^2). \tag{2.14}$$

Therefore, this resolvent indeed belongs to $\mathcal{M}$, and it is easy to see that it is unique in this subclass. In general, the existence of an inverse in the space of Schwartz distributions with respect to the variables x and x' follows from the Hörmander theory, but the necessary properties with respect to the variable q, as well as the uniqueness statement, must be proved. This goes beyond the frame of this paper. We therefore assume here that the potential $u(x)$ is a smooth function rapidly decaying at infinity; the resolvent therefore exists and is unique in $\mathcal{M}$. The class of such potentials is not empty because imposing the same small norm assumption used in [14] is sufficient to prove the existence of the Jost solutions. Moreover, we know (see [16]) that the resolvent in fact exists in much more complicated situations, for instance, when the potential has a nontrivial behavior at infinity.

To demonstrate that the extended resolvent generalizes all notions of the standard approach, we mention that it is directly related to the Green's function of operator (2.7). Indeed, if we introduce

$$G(x, x', \mathbf{k}) = e^{(x-x')\ell_{\Im}(\mathbf{k})}M(x, x'; \ell_{\Im}(\mathbf{k})), \tag{2.15}$$

then it is easy to verify that

$$\mathcal{L}_x G(x, x'; \mathbf{k}) = \delta(x - x'). \tag{2.16}$$

We already mentioned that $L(p; \mathbf{q})$ is an analytic (polynomial) function of $\mathbf{q}$. Taking into account that the standard product of any two operators in terms of the $(p, \mathbf{q})$ variables takes the form (cf. (2.4))

$$(AB)(p; \mathbf{q}) = \int dp' A(p - p'; \mathbf{q} + p')B(p'; \mathbf{q}), \tag{2.17}$$

we see that $M(p;\mathbf{q})$ must in some sense be close to an analytic function of the variable $\mathbf{q}$. As shown in [15, 16], studying its lack of analyticity leads to the notion of the Jost solution. To be more precise, we introduce the operator $\bar{\partial}_2 M_0$ whose kernel in the $(p,\mathbf{q})$ representation is equal to $\partial M_0(p;\mathbf{q})/\partial\bar{\mathbf{q}}_2$. Then it is easy to see that

$$(\bar{\partial}_2 M_0)(x,x';q) = \frac{1}{8\pi|q_1|}\exp\left\{-i\ell_{\Re}\left(\frac{q_2}{2q_1}+iq_1\right)(x-x')\right\}. \tag{2.18}$$

It was shown in [16] that the lack of analyticity of the resolvent M is given in terms of two special reductions of the (truncated) resolvent itself. These operators are defined as

$$|\nu\rangle(x,x';q) = 2\delta(x_2-x_2')\int dq_2\Big((ML_0)\bar{\partial}_2 M_0\Big)(x,x';q), \tag{2.19}$$

$$\langle\omega|(x,x';q) = 2\delta(x_2-x_2')\int dq_2\Big((\bar{\partial}_2 M_0)(L_0 M)\Big)(x,x';q). \tag{2.20}$$

That they are reduced values is more explicit in the $(p,\mathbf{q})$ representation: for example, $|\nu\rangle(p,\mathbf{q}) = (ML_0)(p,\mathbf{q})|_{\mathbf{q}_2=\mathbf{q}_1^2}$. Both kernels $|\nu\rangle(p,\mathbf{q})$ and $\langle\omega|(p,\mathbf{q})$ are independent of $\mathbf{q}_2$ and are analytic functions of $\mathbf{q}_1$ in the upper and lower half-planes with a possible discontinuity at the real $\mathbf{q}_1$ axis. Both tend to $\delta(p)$ as $\mathbf{q}_1\to\infty$. We call these operators the Jost solutions of the NS equation (and its dual), because if we introduce the functions

$$\chi(x,\mathbf{k}) = \int dx' e^{i\mathbf{k}_{\Re}(x_1-x_1')}|\nu\rangle(x,x';\mathbf{k}_{\Im}), \tag{2.21}$$

$$\xi(x',\mathbf{k}) = \int dx e^{i\mathbf{k}_{\Re}(x_1-x_1')}\langle\omega|(x,x';\mathbf{k}_{\Im}) \tag{2.22}$$

for some complex $\mathbf{k}$, then the standard Jost solutions are given by

$$\Phi(x,\mathbf{k}) = e^{-i\ell(\mathbf{k})x}\chi(x;\mathbf{k}), \qquad \Psi(x,\mathbf{k}) = e^{i\ell(\mathbf{k})x}\xi(x;\mathbf{k}). \tag{2.23}$$

We emphasize that Φ and Ψ are the Jost solutions of the original operator (2.7), not of the extended one, i.e., they satisfy

$$\mathcal{L}_x\Phi(x,\mathbf{k}) = 0, \qquad \mathcal{L}_x^{\mathrm{dual}}\Psi(x,\mathbf{k}) = 0, \tag{2.24}$$

while the extension of the operator L causes the Jost solutions to be determined by reductions (2.19) and (2.20) as functions of the complex spectral parameter $\mathbf{k}$. Precisely,

$$i\partial_{x_2}\chi(x,\mathbf{k}) + \partial_{x_1}^2\chi(x,\mathbf{k}) = u(x)\chi(x,\mathbf{k}) + 2i\mathbf{k}\partial_{x_1}\chi(x,\mathbf{k}), \tag{2.25}$$

and it is easy to verify that by the above definitions, $\chi(x,\mathbf{k})$ is normalized to 1 at $\mathbf{k}\to\infty$.

Operators (2.19) and (2.20) satisfy the "differential equations"

$$L|\nu\rangle = |\nu\rangle L_0, \qquad \langle\omega|L = L_0\langle\omega|, \tag{2.26}$$

which are of course equivalent to Eqs. (2.24) after transformations (2.21), (2.22), and (2.23). The reality condition for the potential $u(x)$ is equivalent to

$$|\nu\rangle^\dagger = \langle\omega|, \tag{2.27}$$

which can also be written as $\overline{\Phi(x,\mathbf{k})} = \Psi(x,\bar{\mathbf{k}})$.

In [18], we also proved that the operators $|\nu\rangle$ and $\langle\omega|$ are the inverses of each other,

$$\langle\omega|\nu\rangle = I, \qquad |\nu\rangle\langle\omega| = I, \tag{2.28}$$

and that we have a bilinear representation of the operator L and of the resolvent M by their means,

$$L = |\nu\rangle L_0 \langle\omega|, \tag{2.29}$$
$$M = |\nu\rangle M_0 \langle\omega|. \tag{2.30}$$

Using the resolvent approach, we thus obtain equations that have the explicit meaning of dressing the zero-potential operator L_0 with the operators $|\nu\rangle$ and $\langle\omega|$. One feature of our approach is that these dressing operators are in turn given as reductions (2.19) and (2.20) of the resolvent itself.

To formulate the inverse problem, we use the analyticity properties of the kernels $|\nu\rangle(p,\mathbf{q})$ and $\langle\omega|(p,\mathbf{q})$ and introduce their boundary values on the real axis from the upper and lower half-planes. In terms of the (x,q) representation, the corresponding kernels are defined by means of the limits

$$|\nu^{\pm}\rangle(x,x';q) = \lim_{q_1\to\pm 0} |\nu\rangle(x,x';q), \tag{2.31}$$
$$\langle\omega^{\pm}|(x,x';q) = \lim_{q_1\to\pm 0} \langle\omega|(x,x';q), \tag{2.32}$$

where the l.h.s.'s, of course, do not depend on q. Because the potential is real, they are mutually conjugate,

$$|\nu^{\pm}\rangle^{\dagger} = \langle\omega^{\mp}| \tag{2.33}$$

(see (2.3) and (2.27)). The spectral data are defined [17, 18] as

$$F = \langle\omega^-|\nu^+\rangle, \tag{2.34}$$

and by (2.28),

$$|\nu^+\rangle = |\nu^-\rangle F, \qquad \langle\omega^-| = F\langle\omega^+|. \tag{2.35}$$

These spectral data have the properties

$$F^{\dagger} = F, \qquad [L_0, F] = 0, \qquad q = 0. \tag{2.36}$$

In terms of the $(p,\mathbf{q})$ representation, the last equality means that there exists a function f of two real variables such that we have the representation

$$F(p;\mathbf{q}) = \delta(p_2 - p_1(p_1 + 2\mathbf{q}_{1\Re}))f(p_1 + \mathbf{q}_{1\Re};\mathbf{q}_{1\Re}), \tag{2.37}$$

and then (2.36) means that $\overline{f(\alpha,\beta)} = f(\beta,\alpha)$. In these terms, we can use (2.21) to rewrite the first equation in (2.35), for instance, in the form

$$\chi^+(x,k) = \int d\alpha \chi^-(x,\alpha) e^{ix(\ell(k)-\ell(\alpha))} f(\alpha,k). \tag{2.38}$$

We thus see that Eqs. (2.35) are equivalent to the standard (see [10, 11]) formulation of the nonlocal Riemann–Hilbert problem for the Jost solutions of the NS equation. In what follows, we assume the unique solvability of these equations.

3 Integrable symmetries and time evolutions

3.1 Deformations of the spectral data

We now consider two potentials: $u(x)$ with the corresponding operators L, M, $|\nu\rangle$, and $\langle\omega|$ and the spectral data F and the potential $\tilde{u}(x)$ with the corresponding $\widetilde{L}$, $\widetilde{M}$, $|\tilde{\nu}\rangle$, $\langle\tilde{\omega}|$, and $\widetilde{F}$. Then the two resolvents are related via (2.28) and (2.30) using "scattering on a nontrivial background" [20],

$$\widetilde{M} = \eta M \eta^{-1}, \tag{3.1}$$

where we introduce an operator η such that

$$\eta = |\widetilde{\nu}\rangle\langle\omega|, \qquad \eta^{-1} = |\nu\rangle\langle\widetilde{\omega}|. \tag{3.2}$$

This operator is unitary; because of (2.27),

$$\eta^{\dagger} = \eta^{-1}; \tag{3.3}$$

its kernel in the $(p, \mathbf{q})$ representation does not depend on $\mathbf{q}_2$, $\eta(p; \mathbf{q}) = \eta(p; \mathbf{q}_1)$; it is analytic for $\mathbf{q}_{1\Im} \neq 0$ and satisfies the asymptotic condition

$$\lim_{\mathbf{q}_1 \to \infty} \eta(p; \mathbf{q}_1) = \delta(p). \tag{3.4}$$

For the boundary values of η on the real $\mathbf{q}_1$ axis (defined by analogy with (2.31) and (2.32)), we obtain

$$\eta^{\pm} = |\widetilde{\nu}^{\pm}\rangle\langle\omega^{\pm}| \tag{3.5}$$

from (3.2), and from (2.35), we then find

$$\eta^{+} - \eta^{-} = |\tilde{\nu}^{-}\rangle(\widetilde{F} - F)\langle\omega^{+}|. \tag{3.6}$$

We suppose that a curve in the spectral-data space is parametrized by t. Correspondingly, all objects M, L, U, $|\nu\rangle$, and $\langle\omega|$ become dependent on t; we therefore have a symmetry of the NS equation. Let the nontilde variables correspond to some value t and the tilde variables correspond to the value $t + \tau$. Then $\eta = \eta(t, \tau)$, and by (2.28) and (3.2), $\eta(t, 0) = I$. We set

$$A = -i \frac{\partial \eta}{\partial \tau}\bigg|_{\tau=0}. \tag{3.7}$$

This operator is Hermitian by (3.3), $A^{\dagger} = A$, and its kernel $A(p, \mathbf{q})$ is analytic in the upper and lower half-planes and by (3.4) satisfies the asymptotic condition

$$\lim_{\mathbf{q}_1 \to \infty} A(p; \mathbf{q}_1) = 0. \tag{3.8}$$

It follows from (3.6) that the boundary values of this operator at the real axis satisfy

$$A^{+} - A^{-} = -i|\nu^{-}\rangle \frac{\partial F}{\partial t}\langle\omega^{+}|. \tag{3.9}$$

It is clear that the analyticity properties of $A(p; \mathbf{q})$, together with (3.8) and (3.9), uniquely determine the operator A in the $(p, \mathbf{q})$ representation via the Cauchy formula

$$A(p; \mathbf{q}) = \frac{-1}{2\pi} \int \frac{d\mathbf{q}'_{1\Re}}{\mathbf{q}'_{1\Re} - \mathbf{q}_1} \left(|\nu^{-}\rangle \frac{\partial F}{\partial t} \langle\omega^{+}| \right)(p; \mathbf{q}'); \tag{3.10}$$

we recall that the kernels of all objects in the r.h.s., $|\nu\rangle$, F, and $\langle\omega|$, depend neither on $\mathbf{q}_2$ nor on $\mathbf{q}_{1\Im}$ in the $(p,\mathbf{q})$ representation. We can use (2.4) to rewrite this kernel in the x representation as

$$A(x,x';q) = -i\,\mathrm{sgn}\,q_1 e^{-q_1(x_1-x_1')}\theta((x_1-x_1')q_1)\left(|\nu^-\rangle\frac{\partial F}{\partial t}\langle\omega^+|\right)(x,x';q), \tag{3.11}$$

where the last factor is independent of q by construction.

If A is known, then for the resolvent and the operator L itself, we have

$$\frac{\partial M}{\partial t} = i[A,M], \qquad \frac{\partial L}{\partial t} = i[A,L] \tag{3.12}$$

from (3.1) and (3.7). Moreover, using (2.28) to rewrite (3.2) as $|\widetilde{\nu}\rangle = \eta|\nu\rangle$ and $\langle\widetilde{\omega}| = \langle\omega|\eta^{-1}$, we obtain the derivatives of the Jost solutions along the curve from (3.7):

$$\frac{\partial|\nu\rangle}{\partial t} = iA|\nu\rangle, \qquad \frac{\partial\langle\omega|}{\partial t} = -i\langle\omega|A. \tag{3.13}$$

The flows of the operator L, the resolvent, and the Jost solutions are determined using the IST and are given by the operator A, i.e., by dressing the flow of the spectral data with (3.10) or (3.11). Taking into account that L_0 is independent of t by definition, we obtain

$$i\frac{\partial U}{\partial t} = [A,L] \tag{3.14}$$

for the potential U from (2.8) and (3.12). Due to (2.10), the kernel of the l.h.s. is independent of $\mathbf{q}$ in the $(p,\mathbf{q})$ representation, and $A(p;\mathbf{q})$ decays as $\mathbf{q}_1\to\infty$. Taking (2.8) and (2.9) into account, we see that commutators with iD_2 and U in the r.h.s. also decay; the nonzero result therefore gives only the commutator with D_1^2,

$$i\frac{\partial U}{\partial t} = -\partial_{x_1}^2 A - 2(\partial_{x_1}A)D_1,$$

where we introduce

$$\partial_{x_j}^n A = \underbrace{[D_j,\ldots,[D_j}_{n\,\text{times}},A]\ldots] \tag{3.15}$$

such that $(\partial_{x_j}A)(x,x',q) = A_{x_j}(x,x',q) + A_{x_j'}(x,x',q)$ in the x representation. Again, $\partial_{x_1}^2 A$ decays as $\mathbf{q}_1\to\infty$; therefore

$$\frac{\partial U}{\partial t} = 2\partial_{x_1}A^{(-1)}, \tag{3.16}$$

where $A^{(-1)}$ is the operator whose kernel $A^{-1}(p,\mathbf{q})$ is equal to the residue of $A(p;\mathbf{q})$ at infinity, i.e.,

$$A^{(-1)}(p;\mathbf{q}) = \lim_{\mathbf{q}_1\to\infty} A(p;\mathbf{q})\mathbf{q}_1. \tag{3.17}$$

From (3.10), we have

$$A^{(-1)}(p;\mathbf{q}) = \frac{1}{2\pi}\int d\mathbf{q}_{1\Re}'\left(|\nu^-\rangle\frac{\partial F}{\partial t}\langle\omega^+|\right)(p;\mathbf{q}') \tag{3.18}$$

and from (3.11),

$$A(x,x';q) = \delta(x_1-x_1')\left(|\nu^-\rangle\frac{\partial F}{\partial t}\langle\omega^+|\right)(x,x';q) \tag{3.19}$$

in the x representation. By construction, the last factor is proportional to $\delta(x_2-x_2')$ and is independent of q. We can therefore see that the r.h.s. of (3.16) is indeed a multiplication operator in the x representation, as it must be according to (2.10).

3.2 Linear symmetries of the spectral data

To obtain integrable evolution equations associated with the NS equation from this construction, we must consider linear symmetries of the spectral data F. We therefore impose the condition that there exist operators $a^{\pm}$ such that

$$i\frac{\partial F}{\partial t} = a^{-}F - Fa^{+}. \tag{3.20}$$

Because of (2.37), we can chose kernels of these operators in the $(p, \mathbf{q})$ representation that are independent of $\mathbf{q}_2$ and $\mathbf{q}_{1\Im}$, i.e., without loss of generality,

$$a^{\pm}(p;\mathbf{q}) = a^{\pm}(p;\mathbf{q}_{1\Re}). \tag{3.21}$$

Properties (2.36) of the spectral data show that we can impose the conditions

$$a^{+} = (a^{-})^{\dagger}, \qquad [L_0, a^{\pm}] = 0, \tag{3.22}$$

again without loss of generality.

Returning to (3.9) and using (2.35) and (3.20), we obtain

$$A^{+} - |\nu^{+}\rangle a^{+}\langle\omega^{+}| = A^{-} - |\nu^{-}\rangle a^{-}\langle\omega^{-}|. \tag{3.23}$$

By construction, $A^{\pm}$, $|\nu^{\pm}\rangle$, and $\langle\omega^{\pm}|$ have analytic continuations in the upper and lower half-planes. It is therefore natural to impose the condition that there exists a function $a(p;\mathbf{q})$ (analytic or meromorphic) in $\mathbf{q}_1$ for $\mathbf{q}_{1\Im} \neq 0$ and independent of $\mathbf{q}_2$ such that

$$a^{\pm}(p;\mathbf{q}) = \lim_{\mathbf{q}_{1\Im}\to\pm 0} a(p;\mathbf{q}). \tag{3.24}$$

We can thus consider different symmetries described by operators a with kernels $a(p;\mathbf{q})$ that have singularities of different types in the complex plane. Here, we mainly consider the case where $a(p;\mathbf{q})$ depends on $\mathbf{q}_1$ polynomially. Because, as is well known, the expansion coefficients at infinity of the Jost solutions of the NS equation are given by recursion relations in terms of the potential u, we call such symmetries *explicit*. In particular, because a has no discontinuity on the real axis for such symmetries, i.e.,

$$a^{-} = a^{+}, \tag{3.25}$$

conditions (3.22) give

$$a = a^{\dagger}, \qquad [L_0, a] = 0. \tag{3.26}$$

We then have

$$i\frac{\partial F}{\partial t} = [a, F] \tag{3.27}$$

instead of (3.20), and (3.10) can be written as

$$A = (|\nu\rangle a\langle\omega|)_{-}, \tag{3.28}$$

$$P = (|\nu\rangle a\langle\omega|)_{+}, \tag{3.29}$$

$$|\nu\rangle a\langle\omega| = A + P, \tag{3.30}$$

where $(\cdot)_{+}$ and $(\cdot)_{-}$ denote positive (polynomial) and negative parts of the $1/\mathbf{q}_1$ expansion at infinity of the kernels in the $(p, \mathbf{q})$ representation. From relations (3.28)–(3.30)

with (2.28) and the commutativity of a with L_0 and with M_0 taken into account, we obtain

$$i\frac{\partial M}{\partial t} = [P, M], \qquad i\frac{\partial L}{\partial t} = [P, L], \tag{3.31}$$

$$i\frac{\partial |\nu\rangle}{\partial t} = P|\nu\rangle - |\nu\rangle a, \qquad i\frac{\partial \langle\omega|}{\partial t} = a\langle\omega| - \langle\omega|P \tag{3.32}$$

instead of (3.12) and (3.13).

It is easy to prove that all polynomials a with properties (3.26) can be obtained as linear combinations of the operators

$$a_{m,n} = \frac{i^m}{2}\{D_1^m, \Delta^n\}, \quad m, n \geq 0, \quad m + n > 0. \tag{3.33}$$

Here,

$$\Delta = X_1 + 2iX_2D_1, \tag{3.34}$$

and we introduce the operators X_j that are operators of multiplication by x_j in the x representation,

$$X_j(x, x'; q) = x_j\delta(x - x'), \quad j = 1, 2. \tag{3.35}$$

It is easy to verify that

$$X_j^\dagger = X_j, \qquad [D_j, X_k] = \delta_{j,k}, \tag{3.36}$$

$$\Delta^\dagger = \Delta, \qquad [D_1, \Delta] = I, \qquad [D_2, \Delta] = 2iD_1, \tag{3.37}$$

$$[L_0, \Delta] = 0. \tag{3.38}$$

Symmetries generated by the operators $a_{m,n}$ were called *additional* in [6]. We here consider time evolutions associated with their subclasses.

3.3 Subsets of commuting symmetries and hierarchies of evolutions

In general, the flows introduced in (3.33) do not commute. On the other hand, it is clear that in the linear span of these operators, there exist subsets of mutually commuting operators. We consider two operators a and a', let t and t' be the corresponding evolution parameters of these symmetries, and let A, A' and P, P' be defined by a and a' as in (3.28) and (3.29). In terms of the spectral data, the commutativity condition for these flows is simply

$$[a, a'] = 0, \tag{3.39}$$

where we assume that a and a' are independent of t and t'. In terms of A and P, this condition is more involved. Indeed, from (3.32),

$$\frac{\partial}{\partial t}\Big(|\nu\rangle a'\langle\omega|\Big) = i\Big[A, |\nu\rangle a'\langle\omega|\Big]$$

and vice versa; we therefore have

$$\frac{\partial}{\partial t}\Big(|\nu\rangle a'\langle\omega|\Big) - \frac{\partial}{\partial t'}\Big(|\nu\rangle a\langle\omega|\Big) = i|\nu\rangle[a, a']\langle\omega| - i[A', A] + i[P', P]. \tag{3.40}$$

In case (3.39), the polynomial and the decreasing parts of this equality can be separated, and we obtain the Zakharov–Shabat conditions

$$\frac{\partial A}{\partial t'} - \frac{\partial A'}{\partial t} = i[A, A'], \qquad \frac{\partial P}{\partial t'} - \frac{\partial P'}{\partial t} = i[P, P']. \tag{3.41}$$

As we see from (3.33) and (3.34), $a_{m,n}$ is a differential operator of order $m+n$. If m and n are not relatively prime, i.e., if there exist numbers $k \neq 1$, m', and n' such that $m = m'k$ and $n = n'k$, then $a_{m,n}$ is equal to $(a_{m',n'})^k$ plus a linear combination of some lower $a_{m'',n''}$, $m''+n'' < m+n$. The linear span of set (3.33) is thus decomposed into the union of subsets of commuting flows. These subsets are labeled by pairs of relatively prime numbers (k,l), the operator of the lowest order in the (k,l) subset is $a_{k,l}$, and all elements of this subset are equal to $(a_{k,l})^m$, $m = 1, 2, \ldots$ (or to their linear combinations). Because these subsets of mutually commuting operators are disjoint, we can consider only symmetries belonging to one (k,l) subset. For simplicity, let a denote $a_{k,l}$. We thus study symmetries determined by the operators a^m, $m = 1, 2, \ldots$, and let the corresponding evolution parameters be denoted by t_m,

$$i\frac{\partial F}{\partial t_m} = [a^m, F], \quad m = 1, 2, \ldots . \tag{3.42}$$

To emphasize the commutativity of these symmetries, we use the term *times* for these parameters.

Using (3.42) and inverse problem (2.35) for the Jost solutions, we can introduce dependence on these times for all objects of the theory; thus, by (3.31) and (3.32),

$$i\frac{\partial M}{\partial t_m} = [P_m, M], \qquad i\frac{\partial L}{\partial t_m} = [P_m, L], \tag{3.43}$$

$$i\frac{\partial |\nu\rangle}{\partial t_m} = P_m|\nu\rangle - |\nu\rangle a^m, \qquad i\frac{\partial \langle\omega|}{\partial t_m} = a^m\langle\omega| - \langle\omega|P_m, \tag{3.44}$$

where by (3.29),

$$P_m = (|\nu\rangle a^m \langle\omega|)_+. \tag{3.45}$$

To be more precise, we mark the original operators with a hat: $\widehat{F}$, $\widehat{U}$, $|\hat{\nu}\rangle$, $\langle\hat{\omega}|$, etc. Then the t-dependent operators $F(t_1, t_2, \ldots)$, etc., are determined by the corresponding initial data,

$$F(t_1, t_2, \ldots)|_{t_1=t_2=\ldots=0} = \widehat{F}, \tag{3.46}$$

etc. We emphasize that all time variables are here introduced in addition to the original variables of the kernels of all operators participating in the construction of the spectral theory of the NS equation. It is clear that the time variables are distinct from the variables of the kernels of operators; in particular, the derivatives with respect to these sets of variables commute.

The times t_m are naturally ordered by powers of the differential operators a^m. We mention that evolutions (3.42) can also be considered if $a(p;\mathbf{q})$ is a meromorphic function in the complex domain of $\mathbf{q}_1$. In this case, of course, P_m stands for the singular part of the Laurent expansion of $|\nu\rangle a^m\langle\omega|$; it is then given in terms of not the potential U but the values of the Jost solutions at the poles of a. It is important that in all these cases *independently of the exact choice of operator* a, evolutions (3.42) are associated with the KP equation and its hierarchy. Indeed, for arbitrary operators a and F, there exists the simple commutator identity

$$[a^3, [a, F]] - \frac{3}{4}[a^2, [a^2, F]] + \frac{1}{4}[a, [a, [a, [a, F]]]] = 0, \tag{3.47}$$

which implies that the spectral data F satisfy the differential equation

$$\frac{\partial^2 F}{\partial t_1 \partial t_3} - \frac{3}{4}\frac{\partial^2 F}{\partial t_2^2} + \frac{1}{4}\frac{\partial^4 F}{\partial t_1^4} = 0, \tag{3.48}$$

i.e., the linearized KPI equation, and so on for higher linearized equations.

Moreover, because of (3.44), we have the operator equations

$$\begin{aligned}
& i\frac{\partial|\nu\rangle}{\partial t_1} = P_1|\nu\rangle - |\nu\rangle a, \qquad (3.49)\\
& i\frac{\partial|\nu\rangle}{\partial t_2} + \frac{\partial^2|\nu\rangle}{\partial t_1^2} = \left(P_2 - P_1^2 - i\frac{\partial P_1}{\partial t_1}\right)|\nu\rangle + 2i\frac{\partial|\nu\rangle}{\partial t_1}a,\\
& i\frac{\partial|\nu\rangle}{\partial t_3} + i\frac{\partial^3|\nu\rangle}{\partial t_1^3} = \left(P_3 + \frac{\partial^2 P_1}{\partial t_1^2} - 2i\frac{\partial P_1}{\partial t_1}P_1 - iP_1\frac{\partial P_1}{\partial t_1} - P_1^3\right)|\nu\rangle +\\
& \qquad + 3i\frac{\partial(P_1|\nu\rangle)}{\partial t_1}a,
\end{aligned}$$

which, in the case where $\partial P_1/\partial t_1 = 0$, are simplified to

$$i\frac{\partial|\nu\rangle}{\partial t_2} + \frac{\partial^2|\nu\rangle}{\partial t_1^2} = \left(P_2 - P_1^2\right)|\nu\rangle + 2i\frac{\partial|\nu\rangle}{\partial t_1}a, \tag{3.50}$$

$$i\frac{\partial|\nu\rangle}{\partial t_3} + i\frac{\partial^3|\nu\rangle}{\partial t_1^3} = \left(P_3 - P_1^3\right)|\nu\rangle + 3iP_1\frac{\partial|\nu\rangle}{\partial t_1}a; \tag{3.51}$$

here, the first equation is the obvious analogue of (2.25) with the operators $P_2 - P_1^2$ and a playing the roles of the potential and of the spectral parameter. Using (3.49) and (3.50), we can rewrite (3.51) as

$$\begin{aligned}
\frac{\partial|\nu\rangle}{\partial t_3} - \frac{3i}{2}\frac{\partial^2|\nu\rangle}{\partial t_1\partial t_2} - \frac{1}{2}\frac{\partial^3|\nu\rangle}{\partial t_1^3} = -i\left(P_3 + \frac{1}{2}P_1^3 - \frac{3}{2}P_1P_2\right)|\nu\rangle -\\
- \frac{3i}{2}P_1\left(i\frac{\partial|\nu\rangle}{\partial t_2} + \frac{\partial^2|\nu\rangle}{\partial t_1^2}\right)a.
\end{aligned}$$

The derivative of this equation with respect to t_1 after $\frac{\partial|\nu\rangle}{\partial t_1}a$ is replaced using (3.50) gives the "dressing" of Eq. (3.48)

$$\begin{aligned}
\frac{\partial^2|\nu\rangle}{\partial t_1\partial t_3} - \frac{3}{4}\frac{\partial^2|\nu\rangle}{\partial t_2^2} + \frac{1}{4}\frac{\partial^4|\nu\rangle}{\partial t_1^4} = \left(\left[P_3 + \frac{1}{2}P_1^3 - \frac{3}{4}\{P_1, P_2\}, P_1\right] + \frac{3i}{4}\frac{\partial P_2}{\partial t_2}\right)|\nu\rangle -\\
- i\left(P_3 + \frac{1}{2}P_1^3 - \frac{3}{2}P_2P_1\right)\frac{\partial|\nu\rangle}{\partial t_1} +\\
+ \frac{3}{4}(P_2 - P_1^2)\left(i\frac{\partial|\nu\rangle}{\partial t_2} + \frac{\partial^2|\nu\rangle}{\partial t_1^2}\right),
\end{aligned} \tag{3.52}$$

where we use (3.41) and $\{\cdot,\cdot\}$ stands for the anticommutator of operators.

The simplest examples of polynomial flows of type (3.42) are given by two special subsets (1,0) and (0,1), which, by (3.33), are respectively

$$a_{1,0} = iD_1, \tag{3.53}$$

$$a_{0,1} = \Delta. \tag{3.54}$$

We consider these evolutions in more detail below.

4 Examples of the simplest evolutions

4.1 Times $t_{m,0}$

We first briefly demonstrate the above technique applied to the standard flows (3.53). The corresponding times and polynomials are denoted by $t_{m,0}$ and $P_{m,0}$. We thus have

$$i\frac{\partial F}{\partial t_{m,0}} = [(iD_1)^m, F], \quad m = 1, 2, \ldots, \tag{4.1}$$

$$i\frac{\partial|\nu\rangle}{\partial t_{m,0}} = P_{m,0}|\nu\rangle - |\nu\rangle(iD_1)^m, \tag{4.2}$$

$$P_{m,0} = (|\nu\rangle(iD_1)^m\langle\omega|)_+. \tag{4.3}$$

Using representation (2.37) for the kernel of operator F, we obtain time evolutions of the spectral data in the standard form

$$\frac{\partial f(\alpha,\beta)}{\partial t_{m,0}} = i[(\alpha^m - \beta^m]f(\alpha,\beta). \tag{4.4}$$

To calculate the polynomials $P_{m,0}$ explicitly, we use the asymptotic expansions of the Jost solutions

$$|\nu\rangle = \sum_{n=0}^{\infty}\nu_n(2iD_1)^{-n}, \qquad \langle\omega| = \sum_{n=0}^{\infty}(2iD_1)^{-n}\omega_n, \tag{4.5}$$

which have the exact meaning in our approach of the $1/\mathbf{q}_1$ expansion at infinity for $|\nu\rangle(p;\mathbf{q})$ and the $1/(\mathbf{q}_1 + p_1)$ expansion for $\langle\omega|(p;\mathbf{q})$. It is known that the coefficients of these expansions depend on the half-plane of $\mathbf{q}_1$, but here we can ignore this fact, referring to [17] and [18] for details. To calculate these coefficients, we can use either the integral equation for the Jost solutions or bilinear relations (2.28) and the relation

$$U = iD_2 + D_1^2 - |\nu\rangle(iD_2 + D_1^2)\langle\omega|, \tag{4.6}$$

which follows from (2.8), (2.9), and (2.29). We then obtain

$$\nu_0 = \omega_0 = I, \tag{4.7}$$

$$\nu_1 = -\omega_1, \qquad \partial_{x_1}\nu_1 = iU, \tag{4.8}$$

$$\omega_2 + \nu_2 = -\nu_1\omega_1 - i\partial_{x_1}(\nu_1 - \omega_1), \tag{4.9}$$

$$\partial_{x_1}\nu_2 = \partial_{x_2}\nu_1 - i\partial_{x_1}^2\nu_1 + \frac{1}{2}\partial_{x_1}\nu_1^2, \tag{4.10}$$

and so on, where notation (3.15) is used. We omit the corresponding recursion relations because we do not need the highest coefficients here. Using these formulas and (4.3), we obtain the combinations involved in (3.49), (3.50), and (3.52):

$$P_{1,0} = iD_1, \tag{4.11}$$

$$P_{2,0} - P_{1,0}^2 = U, \tag{4.12}$$

$$P_{3,0} + \frac{1}{2}P_{1,0}^3 - \frac{3}{2}P_{2,0}P_{1,0} = \frac{3i}{4}(\partial_{x_1}U - \partial_{x_2}\nu_1). \tag{4.13}$$

For the first expression in (3.52), we have

$$\left[P_{3,0} + \frac{1}{2}P_{1,0}^3 - \frac{3}{4}\{P_{2,0}, P_{1,0}\}, P_{1,0}\right] + \frac{3i}{4}\frac{\partial P_{2,0}}{\partial t_{2,0}} = \frac{3i}{4}\left(\frac{\partial U}{\partial t_{2,0}} - \partial_{x_2}U\right). \tag{4.14}$$

By (3.49) and (3.50), $|\nu\rangle$ thus satisfies the equations

$$\frac{\partial|\nu\rangle}{\partial t_{1,0}} = \partial_{x_1}|\nu\rangle, \tag{4.15}$$

$$i\frac{\partial|\nu\rangle}{\partial t_{2,0}} + \frac{\partial^2|\nu\rangle}{\partial t_{2,0}^2} = U|\nu\rangle - 2\frac{\partial|\nu\rangle}{\partial t_{1,0}}D_1, \tag{4.16}$$

which, due to (2.8), (2.9), (2.26), (3.15), and (4.15), are exactly

$$\frac{\partial|\nu\rangle}{\partial t_{2,0}} = \partial_{x_2}|\nu\rangle. \tag{4.17}$$

Correspondingly, from (4.5) and (4.8), we obtain the equalities for the potential U

$$\frac{\partial U}{\partial t_{1,0}} = \partial_{x_1}U, \qquad \frac{\partial U}{\partial t_{2,0}} = \partial_{x_2}U. \tag{4.18}$$

Using representations (2.10) and (2.21), we obtain

$$\frac{\partial\chi(x,k)}{\partial t_{1,0}} = \chi_{x_1}(x,k), \qquad \frac{\partial\chi(x,k)}{\partial t_{2,0}} = \chi_{x_2}(x,k), \tag{4.19}$$

$$\frac{\partial u(x)}{\partial t_{1,0}} = u_{x_1}(x), \qquad \frac{\partial u(x)}{\partial t_{2,0}} = u_{x_2}(x). \tag{4.20}$$

These equalities are often considered the reason to identify $t_{1,0} = x_1$ and $t_{2,0} = x_2$. Here, as mentioned in discussing (3.43)–(3.46), we write

$$\begin{aligned} &\chi(t_{1,0}, t_{2,0}|x_1, x_2, k) = \widehat{\chi}(x_1 + t_{1,0}, x_2 + t_{2,0}, k), \\ &u(t_{1,0}, t_{2,0}|x_1, x_2) = \widehat{u}(x_1 + t_{1,0}, x_2 + t_{2,0}). \end{aligned} \tag{4.21}$$

Now, we see that due to (4.18), expression (4.14) is equal to zero; (3.52) is therefore reduced to

$$\begin{aligned} \frac{\partial^2|\nu\rangle}{\partial t_{1,0}\partial t_{3,0}} - \frac{3}{4}\frac{\partial^2|\nu\rangle}{\partial t_{2,0}^2} + \frac{1}{4}\frac{\partial^4|\nu\rangle}{\partial t_{1,0}^4} = {} & \frac{3}{4}\left(\partial_{x_1}U - \partial_{x_2}\nu_1\right)\frac{\partial|\nu\rangle}{\partial t_{1,0}} + \\ & + \frac{3U}{4}\left(i\frac{\partial|\nu\rangle}{\partial t_{2,0}} + \frac{\partial^2|\nu\rangle}{\partial t_{1,0}^2}\right); \end{aligned} \tag{4.22}$$

and from (4.5) and (4.8), we obtain the standard KPI equation

$$\frac{\partial^2 U}{\partial t_{1,0}\partial t_{3,0}} - \frac{3}{4}\frac{\partial^2 U}{\partial t_{2,0}^2} + \frac{1}{4}\frac{\partial^4 U}{\partial t_{1,0}^4} = \frac{3}{4}\frac{\partial^2 U^2}{\partial t_{1,0}^2}. \tag{4.23}$$

4.2 Times $t_{0,m}$

In this section, we investigate evolutions determined by (3.54). The corresponding times and polynomials are denoted by $t_{0,m}$ and $P_{0,m}$, and we again emphasize that derivatives with respect to $t_{0,m}$ commute with the x variables and the times $t_{m,0}$ considered above are not switched on now. Here, we thus have

$$i\frac{\partial F}{\partial t_{0,m}} = [\Delta^m, F], \quad m = 1, 2, \ldots, \tag{4.24}$$

$$i\frac{\partial|\nu\rangle}{\partial t_{0,m}} = P_{0,m}|\nu\rangle - |\nu\rangle\Delta^m, \tag{4.25}$$

$$P_{0,m} = (|\nu\rangle\Delta^m\langle\omega|)_+, \tag{4.26}$$

where the operator Δ is defined in (3.34). Time evolutions of the standard spectral data (2.37) are now

$$\frac{\partial f(\alpha,\beta)}{\partial t_{0,n}} = i\left[\left(-i\frac{\partial}{\partial\alpha}\right)^n - \left(i\frac{\partial}{\partial\beta}\right)^n\right] f(\alpha,\beta), \tag{4.27}$$

and for simplicity in what follows, we consider only $m = 1, 2, 3$.

The polynomials $P_{0,m}$ can be explicitly written using (4.5), (4.7), and (4.10); it must be taken into account that X_2 commutes with $|\nu\rangle$ and $\langle\omega|$ because of (2.19), (2.20), and (3.35). For the combinations involved in (3.49), (3.50), and (3.52), we thus obtain

$$P_{0,1} = \Delta, \tag{4.28}$$

$$P_{0,2} - P_{0,1}^2 = 4X_2^2 U, \tag{4.29}$$

$$P_{0,3} + \frac{1}{2}P_{0,1}^3 - \frac{3}{2}P_{0,2}P_{0,1} = 6iX_2^3(\partial_{x_1}U - \partial_{x_2}\nu_1) - 6iX_2^2\partial_{x_1}(X_1\nu_1), \tag{4.30}$$

and for the first expression in (3.52), we have

$$\begin{aligned}\left[P_{0,3} + \frac{1}{2}P_{0,1}^3 - \frac{3}{4}\{P_{0,2}, P_{0,1}\}, P_{0,1}\right] + \frac{3i}{4}\frac{\partial P_{0,2}}{\partial t_{0,2}} = \\ = 3iX_2^2\left(\frac{\partial U}{\partial t_{0,2}} - 4X_2[\partial_{x_1}(X_1U) + \partial_{x_2}(X_2U)]\right).\end{aligned} \tag{4.31}$$

Then, by (3.49) and (3.50),

$$\frac{\partial|\nu\rangle}{\partial t_{0,1}} = -i[\Delta, |\nu\rangle] = -i[X_1, |\nu\rangle] + 2X_2\partial_{x_1}|\nu\rangle, \tag{4.32}$$

$$i\frac{\partial|\nu\rangle}{\partial t_{0,2}} + \frac{\partial^2|\nu\rangle}{\partial t_{0,1}^2} = 2i\frac{\partial|\nu\rangle}{\partial t_{0,1}}\Delta + 4X_2^2U|\nu\rangle, \tag{4.33}$$

and the second equation, i.e., the NS equation with respect to these times, is thus the exact analogue of (4.16) with the role of the potential played by the multiplication operator (cf. (2.10))

$$W = 4X_2^2U, \qquad W(x, x', q) = w(x)\delta(x - x'), \qquad w(x) = 4x_2^2u(x). \tag{4.34}$$

Expanding these equations by (4.5) and using (4.7) and (4.8), we derive

$$\frac{\partial U}{\partial t_{0,1}} = 2X_2\partial_{x_1}U, \qquad \frac{\partial U}{\partial t_{0,2}} = 4X_2[\partial_{x_1}(X_1U) + \partial_{x_2}(X_2U)], \tag{4.35}$$

where we take $[X_1, |\nu\rangle] \sim D_1^{-2}$ into account.

The evolution with respect to $t_{0,3}$ via (3.52) is again simplified because the expression in (4.31) is equal to zero due to (4.35). From (4.29) and (4.30), we then obtain

$$\begin{aligned}\frac{\partial^2|\nu\rangle}{\partial t_{0,1}\partial t_{0,3}} - \frac{3}{4}\frac{\partial^2|\nu\rangle}{\partial t_{0,2}^2} + \frac{1}{4}\frac{\partial^4|\nu\rangle}{\partial t_{0,1}^4} = 6X_2^2\left(X_2(\partial_{x_1}U - \partial_{x_2}\nu_1) - \partial_{x_1}(X_1\nu_1)\right)\frac{\partial|\nu\rangle}{\partial t_{0,1}} + \\ + 3X_2^2U\left(i\frac{\partial|\nu\rangle}{\partial t_{0,2}} + \frac{\partial^2|\nu\rangle}{\partial t_{0,1}^2}\right).\end{aligned} \tag{4.36}$$

Using the above relations, we can extract the term of order D_1^{-1}. For the potential W in Eq. (4.33), we then obtain exactly the KPI equation in terms of the times $t_{0,1}$, $t_{0,2}$, and $t_{0,3}$:

$$\frac{\partial^2 W}{\partial t_{0,1}\partial t_{0,3}} - \frac{3}{4}\frac{\partial^2 W}{\partial t_{0,2}^2} + \frac{1}{4}\frac{\partial^4 W}{\partial t_{0,1}^4} = \frac{3}{4}\frac{\partial^2 W^2}{\partial t_{0,1}^2}. \tag{4.37}$$

4.3 Transformation of dependent and independent variables

As shown in the previous section, taking the time $t_{0,m}$ into account gives rise to a new dependent variable $w(x)$ via (4.34). Because of (4.37), this function of five variables, $w(t_{0,1}, t_{0,2}, t_{03}|x)$, satisfies the KPI equation with respect to the $t_{0,m}$ variables,

$$\frac{\partial^2 w}{\partial t_{0,1}\partial t_{0,3}} - \frac{3}{4}\frac{\partial^2 w}{\partial t_{0,2}^2} + \frac{1}{4}\frac{\partial^4 w}{\partial t_{0,1}^4} = \frac{3}{4}\frac{\partial^2 w^2}{\partial t_{0,1}^2}, \tag{4.38}$$

for all x, and Eq. (4.35), together with (2.10) and (4.34), implies

$$\frac{\partial w(x)}{\partial t_{0,1}} = 2x_2 w_{x_1}(x), \qquad \frac{\partial w(x)}{\partial t_{0,2}} = 4x_2[x_1 w(x)_{x_1} + x_2 w(x)_{x_2}], \tag{4.39}$$

which suggests a transformation of the independent variables as well. Indeed, let

$$z_1 = \frac{x_1}{2x_2}, \qquad z_2 = \frac{-1}{4x_2}; \tag{4.40}$$

then instead of (4.39), we obtain

$$\frac{\partial w(x)}{\partial t_{0,1}} = w_{z_1}(x), \qquad \frac{\partial w(x)}{\partial t_{0,2}} = w_{z_2}(x). \tag{4.41}$$

This means that $t_{0,1}$ and $t_{0,2}$ shift the variables z_1 and z_2. In other words, if we introduce

$$\varphi(t_{0,1}, t_{0,2}|z) = w(t_{0,1}, t_{0,2}|x), \tag{4.42}$$

then (cf. (3.46))

$$\varphi(t_{0,1}, t_{0,2}|z) = \widehat{\varphi}(z_1 + t_{0,1}, z_2 + t_{0,2});$$

letting $\widehat{w}$ denote the value of w at $t_{0,1} = t_{0,2} = 0$, we obtain

$$\widehat{w}(x) = \widehat{\varphi}\left(\frac{x_1}{2x_2}, \frac{-1}{4x_2}\right) \tag{4.43}$$

and

$$w(t_{0,1}, t_{0,2}|x) = \widehat{w}\left(\frac{x_1 + 2x_2 t_{0,1}}{1 - 4x_2 t_{0,1}}, \frac{x_2}{1 - 4x_2 t_{0,1}}\right). \tag{4.44}$$

We see that the class of initial data and, correspondingly, the class of solutions of the KPI equation can be essentially extended in this way. Indeed, we seek a solution $\varphi(t_{0,1}, t_{0,2}, t_{0,3})$ of Eq. (4.38) that satisfies the initial data

$$\varphi(t_{0,1}, t_{0,2}, 0) = \widehat{\varphi}(t_{0,1}, t_{0,2}). \tag{4.45}$$

Let

$$w(t_{0,1}, t_{0,2}, t_{0,3}|x) = \varphi(t_{0,1} + z_1, t_{0,2} + z_2, t_{0,3}), \tag{4.46}$$

where z is given by (4.40). This function also solves (4.38), and by (4.45), Eq. (4.43) gives $\widehat{w}(x) = w(0,0,0|x)$. Switching on the times $t_{0,m}$ and constructing the function $u(t_{0,1}, t_{0,2}, t_{0,3}|x)$ with the above procedure, we conclude from (4.34) and (4.43) that it satisfies

$$\widehat{u}(x) = \frac{1}{4x_2^2}\widehat{\varphi}\left(\frac{x_1}{2x_2}, \frac{-1}{4x_2}\right). \tag{4.47}$$

Transforming the function $w(t_{0,1}, t_{0,2}, t_{0,3}|x) = 4x_2^2 u(t_{0,1}, t_{0,2}, t_{0,3}|x)$ with the transformation inverse to (4.40) and setting $z = 0$, we obtain the solution of the Cauchy problem for $\varphi(t_{0,1}, t_{0,2}, t_{0,3})$ from (4.45). By (4.47), the equation is thus solved for the initial data

$$\widehat{\varphi}(t_{0,1}, t_{0,2}) = \frac{1}{4t_{0,2}^2}\widehat{u}\left(\frac{-t_{0,1}}{2t_{0,2}}, \frac{-1}{4t_{0,2}}\right),$$

where $\widehat{u}$ as function of its arguments—not $\widehat{\varphi}$— must satisfy the smoothness condition, the decay-at-spatial-infinity condition, and the small norm assumption in order to guaranty the applicability of the IST.

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References

[1] B. B. Kadomtsev and V. I. Petviashvili, *Sov. Phys. Dokl.* **192** (1970): 539.

[2] V. E. Zakharov and A. B. Shabat, *Funct. Anal. Appl.* **8** (1974): 226.

[3] V. E. Zakharov and A. B. Shabat, *Funct. Anal. Appl.* **13** (1979): 166.

[4] V. E. Zakharov, "On the Dressing Method," in: *Inverse Problems in Action*, ed. P. S. Sabatier (Berlin: Springer, 1990), 602.

[5] E. Date, M. Jimbo, M. Kashiwara, and T. Miwa, "Transformation Groups for Soliton Equations," in: *Nonlinear Integrable Systems: Classical Theory and Quantum Theory,* ed. M. Jimbo and T. Miwa, (Singapore: World Scientific, 1983).

[6] A. Yu. Orlov and E. I. Schulman, *Lett. Math. Phys.* **12** (1986): 171.

[7] A. Yu. Orlov, "Vertex Operator, $\bar{\partial}$-Problem, Symmetries, Variational Identities, and Hamiltonian Formalism for (2+1)-Dimensional Integrable Equations," in: *Proc. Intl. Workshop "Plasma Theory and Nonlinear and Turbulent Processes in Physics"*, Vol. 1, ed. V.G.Bar'yakhtar, V. M. Chernousenko, N. S. Erokhin, A. G. Sitenko, and V. E. Zakharov (Singapore: World Scientific, 1988), 116.

[8] A. S. Fokas and V. E. Zakharov, *J. Nonlinear Sci.* **2** (1992): 109.

[9] B. G. Konopelchenko, *Solitons in Multidimensions* (Singapore: World Scientific, 1993).

[10] V. E. Zakharov and S. V. Manakov, *Sov. Sci. Rev. – Phys. Rev.* **1** (1979): 133.

[11] S. V. Manakov, *Physica* **D3** (1981): 420.

[12] A. S. Fokas and M. J. Ablowitz, *Stud. Appl. Math.* **69** (1983): 211.

[13] M. Boiti, J. Léon, and F. Pempinelli, *Phys. Lett.* **A141** (1989): 96.

[14] Xin Zhou, *Commun. Math. Phys.* **128** (1990): 551.

[15] M. Boiti, F. Pempinelli, A. K. Pogrebkov, and M. C. Polivanov, *Theor. Math. Phys.* **93** (1992): 1200.

[16] M. Boiti, F. Pempinelli, A. K. Pogrebkov, and M. C. Polivanov, *Inverse Problems* **8** (1992): 331.

[17] M. Boiti, F. Pempinelli, and A. Pogrebkov, *Inverse Problems* **10** (1994): 505.

[18] M. Boiti, F. Pempinelli, and A. Pogrebkov, *J. Math. Phys.* **35** (1994): 4683.

[19] M. Boiti, F. Pempinelli, and A. Pogrebkov, *Theor. Math. Phys.* **99** (1994): 511.

[20] M. Boiti, F. Pempinelli, and A. Pogrebkov, *Physica* **D87** (1995): 123.